Mikrokanonski ansambl

$$f(\vec{p},\vec{\Sigma}) = \begin{cases} \dfrac{1}{\Delta\Gamma^*(E)} & E \le \mathcal{H}(\vec{p},\vec{\Sigma}) \le E+\Delta E \\ 0 & \text{van} \end{cases}$$

$$\Gamma^*(E) = \frac{1}{N!\,h^{3N}} \; \Gamma(E)$$

$$\Omega^*(E) = \frac{1}{N!\,h^{3N}} \; \frac{\partial\Gamma(E)}{\partial E} = \frac{\partial\Gamma^*(E)}{\partial E}$$

(gustina stanja) (broj mikrostanja po jediničnom intervalu energije)

$$\Omega^*(E) = \frac{1}{N!\,h^{3N}} \int_{\mathcal{H}=E} \frac{d\Sigma}{|\text{grad}\,\mathcal{H}|}$$

F-ja raspodele u klasičnom limesu

$$f(\vec{p},\vec{\Sigma}) = \frac{1}{\Omega^*(E)} \; \delta\left(\mathcal{H}(\vec{p},\vec{\Sigma}) - E\right)$$

$$\Delta E \to 0$$

$$\langle B \rangle = \frac{1}{\Omega(E)} \int_{\mathcal{H}=E} \frac{B(\vec{p},\vec{\Sigma})}{|\text{grad}\,\mathcal{H}|} \, d\Sigma$$

Normalni sistemi u statističko-termodinamičkom smislu:

$$S(E) = k \ln \Omega^*(E) \approx k \ln \Gamma^*(E)$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,V} \qquad \frac{P}{T} = \left(\frac{\partial S(E)}{\partial V}\right)_{E,N}$$

1. Čestica se nalazi u zapremini V, izvan polja sila. Naći gustinu stanja na energijskoj hiperpovrši $H(p_i, q_i, V) = E$. Primeniti dobijeni rezultat na sistem od N slobodnih elektrona koji se nalazi na temperaturi $T = 0°K$, znajući da su u tom slučaju popunjeni svi nivoi energije od nultog do nekog maksimalnog energijskog nivoa E_F (Fermijeva energija, definisana relacijom $\int_0^{E_F} \Omega^*(E)\, dE = N$). Vodeći računa o degeneraciji nivoa zbog spina elektrona, naći jednoelektronsku gustinu stanja $\Omega^*(E)$ u f-ji E_F i N, smatrajući da elektroni obrazuju idealan gas.

Slobodna čestica $H = \frac{p^2}{2m}$

$$\Gamma^*(E) = \int_{H \le E} d\Gamma^* \quad ; \quad d\Gamma^* = \frac{d\vec{p}\, d\vec{q}}{h^3} \qquad N=1$$

$$\Gamma^*(E) = \frac{V}{h^3} \underbrace{\int d\vec{p}}_{V'}$$

$$H = \frac{p^2}{2m} = E \Rightarrow p^2 = 2mE \Rightarrow R = \sqrt{2mE}$$

$$\int d\vec{p} = 4\pi \int_0^{\sqrt{2mE}} p^2 dp = \frac{4}{3}\pi \left(\sqrt{2mE}\right)^3$$

$$\Gamma^*(E) = \frac{V}{h^3} \cdot \frac{4}{3}\pi \left(\sqrt{2mE}\right)^3$$

$$\Omega^*(E) = \frac{\partial\Gamma(E)}{\partial E} = \frac{4}{3}\frac{\pi V}{h^3}(2m)^{\frac{3}{2}} \frac{3}{2} E^{\frac{1}{2}}$$

$$= \frac{2\pi V}{h^3}(2m)^{\frac{3}{2}} E^{\frac{1}{2}}$$

Što se tiče elektrona, ukupan broj stanja $\Gamma(E)$ množimo sa 2:

$$\Omega^*(E_{El}) = \frac{4\pi V}{h^3}\left(2m\right)^{\frac{3}{2}} E_{El}^{1/2} \qquad (*)$$

Uslov za fermione glasi:

$$\int_0^{E_F} \Omega^*(E_{El})\, dE_{El} = N. \qquad (**)$$

Smenom $(*)$ u $(**)$, posle integraljenja dobijamo

$$\Omega(E_{El}) = \frac{3}{2}\frac{N E_{El}^{1/2}}{E_F^{3/2}}$$

2 Sistem se sastoji od N nezavisnih oscilatora, frekvence oscilovanja ν. Tretirajući ovaj sistem klasično, naći entropiju sistema, S, kao i energiju sistema, E, u f-ji temperature T.

Već je poznata fazna zapremina koja odgovara ovom fizičkom sistemu

$$\Gamma(E) = \frac{1}{N!}\left(\frac{E}{\nu}\right)^N$$

Broj mikrostanja unutar te fazne zapremine dobijamo kao

$$\Gamma^*(E) = \frac{1}{N!\,h^N}\,\Gamma(E) = \frac{1}{(N!)^2}\left(\frac{E}{h\nu}\right)^N$$

$$S = k\ln \Gamma^*(E)$$

$$\vdots$$

$$S = Nk\ln\frac{E}{h\nu} - 2k\left(N\ln N - N\right)$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,V} = \dots = \frac{Nk}{E} \Rightarrow E = NkT$$

3. Sistem se sastoji od N molekula idealnog gasa u zapremini V. Naći entropiju sistema S, kao i energiju sistema, E, u funkciji temperature T.

Fazna zapremina koja odgovara ovom fizičkom sistemu se zna od ranije:

$$\Gamma(E,V) = V^N \frac{\pi^{3N/2}}{\Gamma\left(\frac{3N}{2}+1\right)} (2mE)^{\frac{3N}{2}}$$

$$\Gamma^*(E,V) = \frac{1}{N!\,h^{3N}} \Gamma(E,V)$$

$$= \frac{V^N}{N!\,h^{3N}} \frac{\pi^{3N/2}}{\Gamma\left(\frac{3N}{2}+1\right)} (2mE)^{\frac{3N}{2}}$$

$$S = k \ln \Gamma^*(E,V) = \ldots$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,V} = \ldots$$

(napomenuti da je ovime nađena kalorička j-na stanja)

Domaći: Naći termičku j-nu stanja uz pomoć obrasca

$$\frac{p}{T} = \left(\frac{\partial S}{\partial V}\right)_{E,N}$$

4. Za trodimenzionalni ultrarelativistički gas koji se sastoji od N čestica, naći entropiju kaloričnu i termičnu j-nu stanja.

$$\Gamma(E) = \frac{1}{(3N)!}\left(\frac{8\pi V E^3}{c^3}\right)^N$$

$$\Gamma^*(E) = \frac{1}{N!\, h^{3N}}\, \Gamma(E)$$

$S = k \ln \Gamma^*(E)$ + Korišćenje Sterling-ovog obrasca.

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,V} = \dots$$ kalorična j-na stanja

$(E = 3NkT)$

$$\frac{p}{T} = \left(\frac{\partial S}{\partial V}\right)_{E,N} = \dots$$ termična j-na stanja

$(pV = NkT)$

5 Kod mikrokanonskog ansambla fazna gustina verovatnoće može se uvesti pomoću Dirakove delta-funkcije:

$$f(q_i, p_i, a) = A \, \delta[\mathcal{H}(q_i, p_i, a) - E]$$

a) Odrediti normalizacioni faktor A

b) Pokazati da se srednja vrednost neke fizičke veličine po ansamblu, računata na osnovi tako uvedene fazne gustine verovatnoće, poklapa sa srednjom vrednošću datom u uvodu:

$$\langle B \rangle = \frac{1}{\Omega(E, a)} \int_{\mathcal{H}=E} \frac{B(\vec{p}, \vec{q}, a)}{|\text{grad}\,\mathcal{H}|} d\Sigma$$

$$f(q_i, p_i, a) = A \, \delta[\mathcal{H}(q_i, p_i, a) - E]$$

$$1 = \int_\Gamma f d\Gamma = A \int_\Gamma \delta(\mathcal{H} - E) \, d\Gamma$$

$$\boxed{d\Gamma = \frac{d\Sigma \, dE}{|\text{grad}\,\mathcal{H}|}}$$

$$1 = A \int_{-\infty}^{+\infty} \left(\int_{\mathcal{H}=E} \frac{d\Sigma}{|\text{grad}\,\mathcal{H}|} \right) \delta(\mathcal{H} - E) \, dE$$

$$1 = A \int_{\mathcal{H}=E} \frac{d\Sigma}{|\text{grad}\,\mathcal{H}|} \text{, odnosno } \forall E$$

$$A = \frac{1}{\Omega(E, a)}$$

Dakle:

$$f(q_i, p_i, a) = \frac{1}{\Omega(E,a)} \delta\left[\mathcal{H}(q_i, p_i, a) - E\right].$$

Neka je data proizvoljna varijabla $B = B(q_i, p_i$

$\langle B \rangle = ?$

Po definiciji:

$$\langle B \rangle = \int_{\Gamma(E)} B(q_i, p_i, a) f(q_i, p_i, a)\, d\Gamma =$$

$$= \int B(q_i, p_i, a) \frac{1}{\Omega(E,a)} \delta\left[\mathcal{H}(q_i, p_i, a) - E\right] d\Gamma$$

$$= \int_{-\infty}^{+\infty} \left(\frac{1}{\Omega(E,a)} \int_{\mathcal{H}=E} B(q_i, p_i, a) \frac{d\Sigma}{|\mathrm{grad}\,\mathcal{H}|} \right) \delta(\mathcal{H} - E)\, dE.$$

Sledi da je

$$\langle B \rangle = \frac{1}{\Omega(E,a)} \int_{\mathcal{H}=E} B(q_i, p_i, a) \frac{d\Sigma}{|\mathrm{grad}\,\mathcal{H}|}, \quad \forall E$$

4. Pokazati da ako je srednja vrednost po vremenu od veličine $A = \frac{\partial \mathcal{H}(q,p,a)}{\partial a}$ jednaka odgovarajućoj srednjoj Γ vrednosti po faznom prostoru (Ergodička hipoteza)

$$\langle A \rangle = \frac{\int_{\mathcal{H}=E} \frac{A\, d\Sigma}{|\mathrm{grad}\,\mathcal{H}|}}{\int_{\mathcal{H}=E} \frac{d\Sigma}{|\mathrm{grad}\,\mathcal{H}|}}$$

da je onda integral $\int_{\mathcal{H}\le E} d\Gamma = \Gamma(E,a)$ invarijantan pri kvazistatičkom adijabatskom procesu u kome se a menja veoma sporo.

$$\langle A \rangle = \frac{1}{\Omega(E,a)} \int_{\mathcal{H}=E} A \frac{d\Sigma}{|\mathrm{grad}\,\mathcal{H}|}$$

$\Gamma(E,a)$; $\mathcal{H} = \mathcal{H}(q,p,a)$

$$d\Gamma = \left(\frac{\partial \Gamma}{\partial E}\right)_a dE + \left(\frac{\partial \Gamma}{\partial a}\right)_E da \qquad (*)$$

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial q} dq + \frac{\partial \mathcal{H}}{\partial p} dp + \frac{\partial \mathcal{H}}{\partial a} da$$

$$dE = \int_0^\tau \frac{d\mathcal{H}}{dt} dt = \int_0^\tau \left(\cancel{\frac{\partial \mathcal{H}}{\partial q}\frac{dq}{dt} + \frac{\partial \mathcal{H}}{\partial p}\frac{dp}{dt}} + \frac{\partial \mathcal{H}}{\partial a}\frac{da}{dt}\right) dt$$

$$dE = \int_0^{\tau} \frac{\partial \mathcal{H}}{\partial a} \frac{da}{dt} dt$$

Ako je brzina kojom se parametar a menja konstantna, onda važi:

$$\frac{da}{dt} = \frac{\Delta a}{\tau} \qquad (**)$$

$$dE = \frac{\Delta a}{\tau} \int_0^{\tau} \frac{\partial \mathcal{H}}{\partial a} dt = \Delta a \langle A \rangle_t$$

Ako je sistem ergodičan onda važi da je

$$dE = \frac{1}{\Omega(E,a)} \int_{\mathcal{H}=E} \frac{A\, d\Sigma}{|\mathrm{grad}\, \mathcal{H}|} \Delta a$$

Dakle, iz $(*)$ $\Rightarrow$

$$d\Gamma = \left(\frac{\partial \Gamma}{\partial E}\right)_a \frac{1}{\Omega(E,a)} \int_{\mathcal{H}=E} \frac{A\, d\Sigma}{|\mathrm{grad}\, \mathcal{H}|} \Delta a + \left(\frac{\partial \Gamma}{\partial a}\right)_E da$$

odnosno, zbog $(**)$

$$d\Gamma = \oint_{\mathcal{H}=E} \frac{A\, d\Sigma}{|\mathrm{grad}\, \mathcal{H}|} \Delta a + \frac{\partial \Gamma}{\partial a} \Delta a \qquad (**x)$$

$$\frac{\partial \Gamma}{\partial a} = ?$$

F-ja raspodele za mikrokanonski ansambl u klasičnom limesu glasi:

$$f(\vec{p}, \vec{q}, a) = \frac{1}{\Omega(E, a)} \delta(\mathcal{H} - E)$$

$$\Omega(E, a) = \int_{\Gamma} \delta(\mathcal{H} - E)\, d\Gamma$$

$$\frac{\partial \Omega}{\partial a} = \int_{\Gamma} \delta'(\mathcal{H} - E) \frac{\partial \mathcal{H}}{\partial a}\, d\Gamma$$

Integracijom po energijama

$$\frac{\partial \Omega}{\partial a} = \int_{-\infty}^{+\infty} \left(\int_{\mathcal{H}=E} \frac{\partial \mathcal{H}}{\partial a} \frac{d\Sigma}{|\operatorname{grad} \mathcal{H}|} \right) \delta'(\mathcal{H} - E)\, dE$$

$$\frac{\partial \Omega}{\partial a} = \int_{-\infty}^{+\infty} \Omega(E, a) \left\langle \frac{\partial \mathcal{H}}{\partial a} \right\rangle \delta'(\mathcal{H} - E)\, dE$$

$$\int_{-\infty}^{+\infty} f(x)\, \delta^{(n)}(x - x_0)\, dx = (-1)^n f^{(n)}(x_0)$$

$$\int_{-\infty}^{+\infty} f(x)\, \delta'(x - x_0)\, dx = -f'(x_0)$$

Onda će biti

$$\frac{\partial^2 \Gamma}{\partial a\, \partial E} = -\frac{\partial}{\partial E} \left[\Omega(E, a) \left\langle \frac{\partial \mathcal{H}}{\partial a} \right\rangle \right]$$

Onda je

$$\frac{\partial\Gamma(E,a)}{\partial a} = - \Omega(E,a) \left\langle \frac{\partial \mathcal{H}}{\partial a} \right\rangle$$

$$\frac{\partial\Gamma(E,a)}{\partial a} = - \int_{\mathcal{H}=E} \frac{\left\langle \frac{\partial \mathcal{H}}{\partial a} \right\rangle d\Sigma}{|\mathrm{grad}\,\mathcal{H}|}$$

$$= - \int_{\mathcal{H}=E} \frac{A\, d\Sigma}{|\mathrm{grad}\,\mathcal{H}|}$$

Kada se ~~ovde~~ ovo uvrsti u (***)

$$d\Gamma(E,a) = 0 \quad \Rightarrow \quad \Gamma(E,a) = \mathrm{const}$$

FAZNA GUSTINA VEROVATNOĆE

$$d\tau = d\vec{p}\, d\vec{q}$$

GEOMETRIJA

$$dw = f\, d\tau$$

$$[f] = \frac{1}{J \cdot s}$$

$$d\Gamma = \frac{d\vec{p}\, d\vec{q}}{N!\, h^{3N}}$$

MIKROSTANJA (BROJ)

$$dw = f\, d\Gamma$$

$$[f] = 1$$

$$f(\vec{p}, \vec{q}) = \frac{1}{\Omega(E)}\, \delta(\mathcal{H} - E)$$

$$[\delta(\mathcal{H} - E)] = \frac{1}{J}$$

$$[\Omega(E)] = s$$

$$[f(\vec{p}, \vec{q})] = \frac{1}{J \cdot s}$$

$$f(\vec{p}, \vec{q}) = \frac{1}{\Omega(E)}\, \delta(\mathcal{H} - E)$$

$$[\delta(\mathcal{H} - E)] = \frac{1}{J}$$

$$[\Omega(E)] = \frac{1}{J}$$

$$[f(\vec{p}, \vec{q})] = 1$$

Koncizno

$$f_G(\vec{p}, \vec{q}) = \frac{1}{\Omega_G(E)}\, \delta(\mathcal{H} - E)$$

$$f_M(\vec{p}, \vec{q}) = \frac{1}{\Omega_M(E)}\, \delta(\mathcal{H} - E)$$

$$f_G(\vec{p}, \vec{q}) = \frac{f_M(\vec{p}, \vec{q})}{N!\, h^{3N}}$$

$$\Omega_M(E) = \frac{\Omega_G(E)}{N!\, h^{3N}}$$

$\equiv \Omega$

Ω^*